

Schubert Classes in the Equivariant K-Theory and Equivariant Cohomology of the Lagrangian Grassmannian

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Abstract

We give positive formulas for the restriction of a Schubert Class to a T -fixed point in the equivariant K-theory and equivariant cohomology of the Lagrangian Grassmannian. Our formulas rely on a result of Ghorpade-Raghavan, which gives an equivariant Gröbner degeneration of a Schubert variety in the neighborhood of a T -fixed point of the Lagrangian Grassmannian.

1 Introduction

Let J be the antidiagonal $2n \times 2n$ matrix whose top n antidiagonal entries are 1's and whose bottom n antidiagonal entries are -1's. Then J defines a nondegenerate skew-symmetric inner product on \mathbb{C}^{2n} by $\langle v, w \rangle = v^t J w$, $v, w \in \mathbb{C}^{2n}$. The Lagrangian Grassmannian LGr_n is defined as the set of all n -dimensional complex subspaces V of \mathbb{C}^{2n} which are isotropic under this inner product, i.e., such that for every $v, w \in V$, $\langle v, w \rangle = 0$. The symplectic group $G = Sp_{2n}(\mathbb{C})$ consists of the invertible $2n \times 2n$ complex matrices which preserve this inner product. Let T and B denote the diagonal and upper triangular matrices of G respectively. The natural action of G on LGr_n is transitive and has a unique B -fixed point e_{id} . Thus LGr_n can be identified with G/P_n , where $P_n \supset B$ is the stabilizer of e_{id} . Let W denote the Weyl group of G with respect to T ($= N_G(T)/T$) and W_{P_n} the Weyl group of P_n . For the G -action on LGr_n , the T -fixed points are precisely the cosets $e_\beta := \beta P_n$, $\beta \in W/W_{P_n}$.

Let B^- denote the lower triangular matrices in G . For $\alpha \in W/W_{P_n}$, the (opposite) Schubert variety X_α is the Zariski closure of $B^- e_\alpha$ in LGr_n . The Schubert variety X_α defines classes $[X_\alpha]_K$ in $K_T^*(LGr_n)$, the T -equivariant K-theory of LGr_n , and $[X_\alpha]_H$ in $H_T^*(LGr_n)$, the T -equivariant cohomology of LGr_n .

The T -equivariant embedding $e_\beta \xrightarrow{i} LGr_n$ induces restriction homomorphisms:

$$K_T^*(LGr_n) \xrightarrow{i_K^*} K_T^*(e_\beta) \quad \text{and} \quad H_T^*(LGr_n) \xrightarrow{i_H^*} H_T^*(e_\beta).$$

The image of an element C of $K_T^*(LGr_n)$ or $H_T^*(LGr_n)$ under restriction to e_β is denoted by $C|_{e_\beta}$. The restrictions $C|_{e_\beta}$, evaluated at all $\beta \in W/W_{P_n}$, determine C uniquely. In this paper we obtain combinatorial formulas for $[X_\alpha]_{\mathbb{K}}|_{e_\beta}$ and $[X_\alpha]_{\mathbb{H}}|_{e_\beta}$. Our formula for $[X_\alpha]_{\mathbb{K}}|_{e_\beta}$ is positive in the sense of [Gr–Ra, Conjecture 5.1], and our formula for $[X_\alpha]_{\mathbb{H}}|_{e_\beta}$ is positive in the sense of [Gra]. A positive formula for $[X_\alpha]_{\mathbb{K}}|_{e_\beta}$ also appears in [Wil], and positive formulas for $[X_\alpha]_{\mathbb{H}}|_{e_\beta}$ appear in [Bil] and [Ike].

The proof of our formulas relies on a result of Ghorpade-Raghavan [Gh–Ra], which gives an explicit equivariant Gröbner degeneration of an open neighborhood of X_α centered at e_β to a reduced union of coordinate spaces. The outline of our proof is virtually the same as that of [Kre3], which derives a similar result as here, but for Schubert varieties in the ordinary Grassmannian. In addition, many of the lemmas of [Kre3] and their proofs carry over with little or no modification.

Our formulas for $[X_\alpha]_{\mathbb{K}}|_{e_\beta}$ and $[X_\alpha]_{\mathbb{H}}|_{e_\beta}$ are expressed in terms of ‘semistandard set-valued *shifted* tableaux’. These objects take the place of the ‘semistandard set-valued tableaux’ in [Kre3]. Semistandard set-valued tableaux were introduced by Buch [Buc], and also appear in [KMY1], [KMY2]. The formula for $[X_\alpha]_{\mathbb{H}}|_{e_\beta}$ can also be expressed in terms of ‘subsets of *shifted* diagrams’, which we introduce in Section 5. These objects take the place of the ‘subsets of Young diagrams’ in [Kre3]. It has come to our attention that Ikeda-Naruse have independently discovered subsets of Young diagrams and subsets of shifted diagrams and used them to express formulas for restrictions of Schubert classes to T -fixed points in the equivariant cohomology of the ordinary and Lagrangian Grassmannians respectively.

2 Semistandard Set-Valued Shifted Tableaux

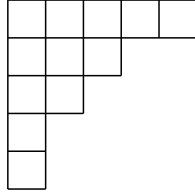
For $k \in \{1, \dots, 2n\}$, define $\bar{k} = 2n + 1 - k$. Let I_n denote the set of all n element subsets $\alpha = \{\alpha(1), \dots, \alpha(n)\}$ of $\{1, \dots, 2n\}$ such that for each $k \in \{1, \dots, 2n\}$, exactly one of k or \bar{k} is in α . We always assume the entries of such a subset are listed in increasing order. For $\alpha \in I_n$, define $\alpha' \in I_n$ by $\alpha' = \{1, \dots, 2n\} \setminus \alpha = \{\bar{\alpha(n)}, \dots, \bar{\alpha(1)}\}$. The map which takes $\{\alpha(1), \dots, \alpha(n)\} \in I_n$ to the permutation $(\alpha(1), \dots, \alpha(n), \bar{\alpha(n)}, \dots, \bar{\alpha(1)}) \in W$ identifies I_n with the set of minimal length coset representatives for W/W_{P_n} . We shall use I_n rather than W/W_{P_n} to index the Schubert varieties and T -fixed points of LGr_n . Fix $\alpha, \beta \in I_n$ for the remainder of this paper.

A **partition** is an ordered list of nonnegative integers $\lambda = (\lambda_1, \dots, \lambda_m)$, $\lambda_1 \geq \dots \geq \lambda_m$. Two partitions are identified if one can be obtained from the other by adding zeros. The **transpose** of λ is the partition $\lambda^t = (\lambda_1^t, \dots, \lambda_p^t)$, where $\lambda_j^t = \#\{i \in \{1, \dots, m\} \mid \lambda_i \geq j\}$, $j = 1, \dots, p$. The partition λ is said to be **symmetric** if $\lambda^t = \lambda$. It is said to be **strict** if $\lambda_i = \lambda_{i+1}$ implies $\lambda_i = 0$, $i = 1, \dots, m$. We denote by L_n (resp. M_n) the set of all symmetric (resp. strict) partitions λ with $\lambda_1 \leq n$.

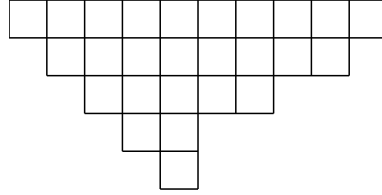
The map π from finite subsets of the positive integers to partitions, given by $\pi : \{\gamma(1), \dots, \gamma(k)\} \mapsto (\gamma(k) - k, \dots, \gamma(1) - 1)$, where $\gamma(1) < \dots < \gamma(k)$, restricts to a bijection from I_n to L_n . The map ρ from partitions to strict partitions, given by $\rho : (\lambda_1, \dots, \lambda_k) \mapsto (\lambda_1, \lambda_2 - 1, \dots, \lambda_l - l + 1)$, where l is maximal such that $\lambda_l - l + 1 \geq 0$, restricts to a bijection from L_n to M_n . We denote the composition $\rho \circ \pi : I_n \rightarrow M_n$ by σ . If $\lambda = \sigma(\alpha)$, then the **length** of α , denoted $l(\alpha)$, is $\lambda_1 + \dots + \lambda_n$.

A **Young diagram** is a collection of boxes arranged into a top and left justified array. A Young diagram is said to be **symmetric** if the length of the i -th row equals the length of the i -th column for all i . To any partition λ we associate the Young diagram D_λ whose i -th row has length λ_i . The j -th column of D_λ has length λ_j^t . Thus λ is symmetric if and only if D_λ is symmetric, and L_n can be identified with the set of all symmetric Young diagrams whose first rows have length $\leq n$.

A **shifted diagram** is a top-justified array of boxes whose left side forms a descending staircase, i.e., the leftmost box of any row is one column to the right of the leftmost box of the row above it. The **length** of a row of a shifted diagram is the number of boxes it contains. To a strict partition λ we associate the shifted diagram \tilde{D}_λ whose i -th row has length λ_i . We call λ the **shape** of \tilde{D}_λ . One sees that M_n can be identified with the set of all shifted diagrams whose first rows have length $\leq n$.



(a) A symmetric Young diagram



(b) A shifted diagram

The bijection $\rho : L_n \rightarrow M_n$ can be viewed in terms of associated partitions. Let λ be a symmetric partition. If we remove all boxes of D_λ which lie below the main diagonal, then we obtain $\tilde{D}_{\rho(\lambda)}$:

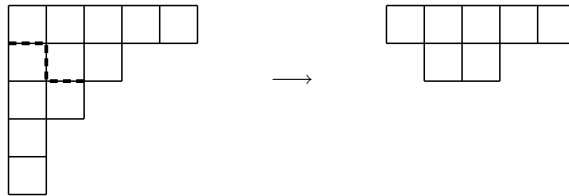


Figure 1: The map ρ

A **set-valued shifted tableau** S is an assignment of a nonempty set of positive integers to each box of a shifted diagram. The entries of S are the positive integers in the boxes. If a positive integer occurs in more than one box of S , then we consider the separate occurrences to be distinct entries. If x is an entry of S , then we define $r(x)$ and $c(x)$ to be the row and column numbers of the box containing x (where the top row is considered the first row and the leftmost column is considered the first column), and we define $z(x)$ to be $x + c(x) - r(x)$. We say that S is a **Young shifted tableau** if each box contains a single entry.

A set-valued shifted tableau is said to be **semistandard** if all entries of any box B are less than or equal to all entries of the box to the right of B and strictly less than all entries of the box below B .

1	2, 3	3	3	4, 6, 7	7, 9
	4	4, 6	6, 7, 8	9, 11	
		8, 10			

Figure 2: A semistandard set-valued shifted tableau

If $\mu = (\mu_1, \dots, \mu_h)$ is any strict partition, then a set-valued shifted tableau S is said to be **on** μ if, for every entry x of S , $x \leq h$ and

$$z(x) \leq \mu_x + x - 1. \quad (1)$$

Example 2.1. Let $\lambda = (2, 1)$, $\mu = (5, 3, 2)$. The following list gives all semistandard set-valued shifted tableaux on μ of shape λ :

1	1								
		1	1						
				1	2				
						2	2		
								3	

Denote the set of semistandard set-valued shifted tableaux on μ of shape λ by $\text{SSV}\tilde{\text{T}}_{\lambda,\mu}$ and the set of semistandard Young shifted tableaux on μ of shape λ by $\text{SSY}\tilde{\text{T}}_{\lambda,\mu}$.

3 Results

Let \mathfrak{t} denote the Lie algebra of T and $R(T)$ the representation ring of T . We have that

$$\begin{aligned} T &= \{\text{diag}(s_1, \dots, s_n, s_n^{-1}, \dots, s_1^{-1}) \mid s_k \in \mathbb{C}^*\} \\ \mathfrak{t} &= \{\text{diag}(s_1, \dots, s_n, -s_n, \dots, -s_1) \mid s_k \in \mathbb{C}\} \\ K_T^*(e_\beta) &\cong R(T) = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \\ H_T^*(e_\beta) &\cong \mathbb{C}[\mathfrak{t}^*] = \mathbb{C}[t_1, \dots, t_n] \end{aligned}$$

For $k = 1, \dots, n$, define $t_{\bar{k}} \in K_T^*(e_\beta)$ to be t_k^{-1} and $t_{\bar{k}} \in H_T^*(e_\beta)$ to be $-t_k$.

Proposition 3.1. *Let $\lambda = \sigma(\alpha)$, $\mu = \sigma(\beta)$. Then*

$$(i) [X_\alpha]_{\kappa}|_{e_\beta} = (-1)^{l(\alpha)} \sum_{S \in \text{SSV}\tilde{\text{T}}_{\lambda,\mu}} \prod_{x \in S} \left(\frac{1}{t_{\beta'(x)} t_{\beta'(z(x))}} - 1 \right).$$

$$(ii) [X_\alpha]_{\mathcal{H}}|_{e_\beta} = \sum_{S \in \text{SSY}\tilde{\text{T}}_{\lambda,\mu}} \prod_{x \in S} (-t_{\beta'(x)} - t_{\beta'(z(x))}).$$

Example 3.2. *Consider LGr_3 , $\alpha = \{1, 3, \bar{2}\}$, $\beta = \{3, \bar{2}, \bar{1}\}$. Then $\sigma(\alpha) = (2)$, $\sigma(\beta) = (3, 2)$, $l(\alpha) = 2$, $\beta' = \{1, 2, \bar{3}\}$. The semistandard set-valued tableaux on $\sigma(\beta)$ of shape $\sigma(\alpha)$ are:*

$$\begin{array}{ccc} \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} & \boxed{2} \\ & & & & & \\ & & \boxed{1} & \boxed{1, 2} & \boxed{1, 2} & \boxed{2} \end{array}$$

Therefore,

$$\begin{aligned} [X_\alpha]_{\kappa}|_{e_\beta} &= \left(\frac{1}{t_1^2} - 1 \right) \left(\frac{1}{t_1 t_2} - 1 \right) + \left(\frac{1}{t_1^2} - 1 \right) \left(\frac{t_3}{t_2} - 1 \right) + \left(\frac{1}{t_2^2} - 1 \right) \left(\frac{t_3}{t_2} - 1 \right) \\ &\quad + \left(\frac{1}{t_1^2} - 1 \right) \left(\frac{1}{t_2^2} - 1 \right) \left(\frac{1}{t_1 t_2} - 1 \right) + \left(\frac{1}{t_1^2} - 1 \right) \left(\frac{1}{t_1 t_2} - 1 \right) \left(\frac{t_3}{t_2} - 1 \right) \\ [X_\alpha]_{\mathcal{H}}|_{e_\beta} &= (-2t_1)(-t_1 - t_2) + (-2t_1)(-t_2 + t_3) + (-2t_2)(-t_2 + t_3). \end{aligned}$$

Remark 3.3. As we shall show in Section 4, each term in the products of Proposition 3.1(i) and (ii) is of the form $e^\theta - 1$ and θ respectively, where θ is a positive root with respect to the Borel subgroup B^- .

4 The Class of a Schubert Variety

The **Plücker map** $LGr_n \rightarrow \mathbb{P}(\wedge^n \mathbb{C}^{2n})$ is defined by $V \mapsto [v_1 \wedge \cdots \wedge v_n]$, where $\{v_1, \dots, v_n\}$ is any basis for V . The Plücker map is a closed immersion, giving LGr_n its projective variety structure.

Reduction to an Affine Variety

Under the Plücker map, e_β maps to $[e_{\beta(1)} \wedge \cdots \wedge e_{\beta(n)}] \in \mathbb{P}(\wedge^n \mathbb{C}^{2n})$. Define p_β to be the homogeneous (**Plücker**) coordinate $[e_{\beta(1)} \wedge \cdots \wedge e_{\beta(n)}]^* \in \mathbb{C}[\mathbb{P}(\wedge^n \mathbb{C}^{2n})]$. Let \mathcal{O}_β be the distinguished open set of LGr_n defined by $p_\beta \neq 0$. Then \mathcal{O}_β is isomorphic to the affine space $\mathbb{C}^{n(n+1)/2}$, with e_β the origin. Indeed, \mathcal{O}_β can be identified with the space of $2n \times n$ complex matrices of the form $K \cdot M$, where K and M are defined as follows:

1. K is the $2n \times 2n$ diagonal matrix which has 1's in the diagonal entries of rows $\beta(1), \dots, \beta(n)$ and rows $n+1, \dots, 2n$, and -1 's in the diagonal entries of all other rows.
2. M is any $2n \times n$ complex matrix for which rows $\beta(1), \dots, \beta(n)$ form the $n \times n$ identity matrix and rows $\beta'(1), \dots, \beta'(n)$ form an $n \times n$ antisymmetric (i.e., symmetric about the antidiagonal) matrix.

Under this identification, we index the rows of \mathcal{O}_β by $\{1, \dots, 2n\}$ and the columns by β . We then choose the coordinates of \mathcal{O}_β to be the matrix elements y_{ab} , where $a \in \beta'$, $b \in \beta$, and $a \leq \bar{b}$ (note: due to the antisymmetry in the matrices M of 2, each matrix element y_{ab} , where $a \in \beta'$, $b \in \beta$, and $a > \bar{b}$ must be plus or minus one of our chosen coordinates). Thus $\{(a, b) \in \beta' \times \beta \mid a \leq \bar{b}\}$, which we denote by \mathfrak{R}_β , forms an indexing set for the coordinates of \mathcal{O}_β .

Example 4.1. Let $n = 4$, $\beta = \{1, 4, \bar{3}, \bar{2}\}$. Then $\beta' = \{2, 3, \bar{4}, \bar{1}\}$ and

$$\mathcal{O}_\beta = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ -y_{21} & -y_{24} & -y_{2\bar{3}} & -y_{2\bar{2}} \\ -y_{31} & -y_{34} & -y_{3\bar{3}} & -y_{3\bar{2}} \\ 0 & 1 & 0 & 0 \\ y_{\bar{4}1} & y_{\bar{4}4} & y_{34} & y_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ y_{\bar{1}1} & y_{\bar{1}4} & y_{31} & y_{21} \end{pmatrix}, y_{ab} \in \mathbb{C} \right\}.$$

The space \mathcal{O}_β is T -stable, and for $\mathbf{s} = \text{diag}(s_1, \dots, s_n, s_n^{-1}, \dots, s_1^{-1}) \in T$ and coordinate functions $y_{ab} \in \mathbb{C}[\mathcal{O}_\beta]$,

$$\mathbf{s}(y_{ab}) = \frac{s_b}{s_a} y_{ab},$$

where $s_k^- := s_k^{-1}$, $k = 1, \dots, n$.

The equivariant embeddings $e_\beta \xrightarrow{j} \mathcal{O}_\beta \xrightarrow{k} LGr_n$ induce homomorphisms

$$K_T^*(LGr_n) \xrightarrow{k^*} K_T^*(\mathcal{O}_\beta) \xrightarrow{j^*} K_T^*(e_\beta).$$

The map j^* is an isomorphism, identifying $K_T^*(\mathcal{O}_\beta)$ with $K_T^*(e_\beta)$. Define $Y_{\alpha,\beta} = X_\alpha \cap \mathcal{O}_\beta$, an affine subvariety of \mathcal{O}_β . We have

$$[X_\alpha]_{\mathbb{K}}|_{e_\beta} = j^* \circ k^*([X_\alpha]_{\mathbb{K}}) = j^*([k^{-1}X_\alpha]_{\mathbb{K}}) = j^*([Y_{\alpha,\beta}]_{\mathbb{K}}) = [Y_{\alpha,\beta}]_{\mathbb{K}}.$$

Applying analogous arguments for equivariant cohomology, we obtain

$$[X_\alpha]_{\mathbb{H}}|_{e_\beta} = [Y_{\alpha,\beta}]_{\mathbb{H}}.$$

Reduction to a Union of Coordinate Subspaces

Let $\lambda = \sigma(\alpha)$, $\mu = \sigma(\beta)$. Let $\text{SV}\tilde{\text{T}}_{\lambda,\mu}$ denote the set of all set-valued shifted tableaux (not necessarily semistandard) of shape λ on μ . For $S \in \text{SV}\tilde{\text{T}}_{\lambda,\mu}$, define

$$W_S = V(\{y_{\beta'(x), \overline{\beta'(z(x))}} \mid x \in S\}),$$

a coordinate subspace of \mathcal{O}_β . Define

$$W_{\alpha,\beta} = \bigcup_{P \in \text{SSY}\tilde{\text{T}}_{\lambda,\mu}} W_P.$$

The following lemma, whose proof is a consequence of [Gh–Ra] and appears in Section 5, reduces the proof of Proposition 3.1 to computing the class of a union of coordinate subspaces.

Lemma 4.2. $[Y_{\alpha,\beta}]_{\mathbb{K}} = [W_{\alpha,\beta}]_{\mathbb{K}}$

Proof of Proposition 3.1. (i) The proofs of Lemmas 4.3 and 4.4 and consequently of Proposition 2.2(i) of [Kre3] carry through if the following modifications are made: (a) the word ‘tableau’ is replaced by ‘shifted tableau’ in all steps and all required definitions, and (b) $t_{\beta(d+1-x)}$ and $t_{\beta'(x+c(x)-r(x))}$ are replaced by $t_{\overline{\beta'(z(x))}}$ and $t_{\beta'(x)}$ respectively wherever they occur. The latter modification accounts for the difference in the definitions of W_S .

(ii) There is a standard ring homomorphism from $K_T^*(\mathcal{O}_\beta)$ to $H_T^*(\mathcal{O}_\beta)$, the Chern character map, given by $ch : t_i \mapsto e^{-t_i} = 1 - t_i + t_i^2/2 - t_i^3/3 + \dots$. If $Y \subset \mathcal{O}_\beta$ is a T -stable subvariety, then

$$ch : [Y]_{\mathbb{K}} \mapsto [Y]_{\mathbb{H}} + \text{higher order terms}.$$

Thus, $[Y_{\alpha,\beta}]_{\mathbb{H}}$ is the lowest order term of

$$(-1)^{l(\alpha)} \sum_{S \in \text{SSY}\tilde{\text{T}}_{\lambda,\mu}} \prod_{x \in S} \left(\frac{1}{e^{-t_{\beta'(x)}} e^{-t_{\beta'(z(x))}}} - 1 \right),$$

which equals

$$\sum_{S \in \text{SSYT}_{\lambda, \mu}} \prod_{x \in S} (-t_{\beta'(x)} - t_{\beta'(z(x))}).$$

□

Proof of Remark 3.3. Let $\eta = \pi(\beta)$, so that $\mu = \sigma(\beta) = \rho(\pi(\beta)) = \rho(\eta)$. One can show that $\eta_j = \#\{i \in \{1, \dots, n\} \mid \beta'(i) < \beta(n+1-j)\}$, $j = 1, \dots, n$. Therefore

$$i \leq \eta_j \iff \beta'(i) < \beta(n+1-j) \quad (2)$$

We look at one term $-t_{\beta'(x)} - t_{\beta'(z(x))}$ in the product of (ii). Substituting $i = z(x)$ and $j = x$ into (2), we obtain: $z(x) \leq \eta_x \iff \beta'(z(x)) < \beta(n+1-x) = \beta'(x) \iff \beta'(x) < \beta'(z(x))$. Since S is on μ , x satisfies (1), i.e., $z(x) \leq \mu_x + x - 1 = \eta_x$. Thus $\beta'(x) < \beta'(z(x))$. In addition, since $x \leq z(x)$, $\beta'(x) \leq \beta'(z(x))$.

Thus $-t_{\beta'(x)} - t_{\beta'(z(x))}$ is of the form $-t_a - t_b$, $a \leq b$, $a < \bar{b}$, and hence $a \leq n$. Clearly this is a positive root if $b \leq n$. If $b > n$, then letting $c = \bar{b}$, we have $-t_a - t_b = -t_a + t_c$, $a < c \leq n$, which is also a positive root. □

5 Four Equivalent Models: $\tilde{\mathcal{F}}''_{\lambda, \mu}$, $\tilde{\mathcal{F}}_{\lambda, \mu}$, $\tilde{\mathcal{D}}_{\lambda, \mu}$, and $\text{SSYT}_{\lambda, \mu}$

In this section we prove Lemma 4.2. We assume all definitions from Sections 5 and 6 of [Kre3]. Let ζ, η be symmetric partitions with $\zeta \leq \eta$. Let D_η be the (symmetric) Young diagram associated to η .

- A family F of nonintersecting paths on D_η is said to be **symmetric** if $(i, j) \in F \iff (j, i) \in F$. In such case, it can be checked inductively that for any path p of F , \bar{p} is also a path of F , where \bar{p} is the path obtained by replacing each (i, j) of p by (j, i) . We define $\overline{\mathcal{F}}_{\zeta, \eta}$ and $\overline{\mathcal{F}}''_{\zeta, \eta}$ to be the set of all symmetric elements of $\mathcal{F}_{\zeta, \eta}$ and $\mathcal{F}''_{\zeta, \eta}$ respectively.
- A subset D of D_η is said to be **symmetric** if $(i, j) \in D \iff (j, i) \in D$. We define $\overline{\mathcal{D}}_{\zeta, \eta}$ to be the set of all symmetric elements of $\mathcal{D}_{\zeta, \eta}$.
- A semistandard tableau P of shape ζ is said to be **symmetric** if $P_{i, j} - i = P_{j, i} - j$ for all $(i, j) \in D_\eta$. We define $\text{SSYT}_{\zeta, \eta}$ to be the set of all symmetric elements of $\text{SSYT}_{\zeta, \eta}$.

By Lemma 5.14 of [Kre3], $\mathcal{F}''_{\zeta, \eta} = \mathcal{F}_{\zeta, \eta}$. Hence $\overline{\mathcal{F}}''_{\zeta, \eta} = \overline{\mathcal{F}}_{\zeta, \eta}$. The bijections $\mathcal{F}_{\zeta, \eta} \rightarrow \mathcal{D}_{\zeta, \eta}$ and $\mathcal{D}_{\zeta, \eta} \rightarrow \text{SSYT}_{\zeta, \eta}$ given in [Kre3] restrict to bijections $\overline{\mathcal{F}}_{\zeta, \eta} \rightarrow \overline{\mathcal{D}}_{\zeta, \eta}$ and $\overline{\mathcal{D}}_{\zeta, \eta} \rightarrow \text{SSYT}_{\zeta, \eta}$ respectively.

Let $\lambda = \rho(\zeta)$, $\mu = \rho(\eta)$, and let \tilde{D}_μ be the shifted diagram associated with μ . We have the notions of subsets of \tilde{D}_μ and families of nonintersecting paths on \tilde{D}_μ , defined analogously as in [Kre3]. If $D \in \overline{\mathcal{D}}_{\zeta, \eta}$, then we define $\rho(D)$ to be the

subset of \tilde{D}_μ obtained by removing all boxes of D below the main diagonal of D . If $F \in \tilde{\mathcal{F}}_{\zeta,\eta}'' = \tilde{\mathcal{F}}_{\zeta,\eta}''$, then we define $\rho(F)$ to be the family of nonintersecting paths on \tilde{D}_μ obtained by removing all boxes in all paths of F below the main diagonal of F . If $P \in \text{SSYT}_{\zeta,\eta}$, then we define $\rho(P)$ to be the semistandard shifted tableau obtained by removing all boxes of P below the main diagonal and their entries. Define $\tilde{\mathcal{D}}_{\lambda,\mu} = \rho(\overline{\mathcal{D}}_{\zeta,\eta})$, $\tilde{\mathcal{F}}_{\lambda,\mu} = \rho(\tilde{\mathcal{F}}_{\zeta,\eta})$, $\tilde{\mathcal{F}}_{\lambda,\mu}'' = \rho(\tilde{\mathcal{F}}_{\zeta,\eta}'')$, and note $\text{SSYT}_{\lambda,\mu} = \rho(\text{SSYT}_{\zeta,\eta})$.

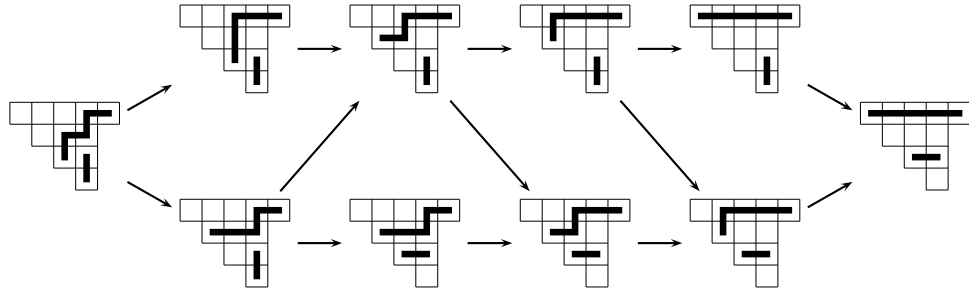
We have that $\tilde{\mathcal{F}}_{\lambda,\mu}'' = \tilde{\mathcal{F}}_{\lambda,\mu}$, and under ρ , the bijections $\tilde{\mathcal{F}}_{\zeta,\eta}'' \rightarrow \overline{\mathcal{D}}_{\zeta,\eta}$ and $\overline{\mathcal{D}}_{\zeta,\eta} \rightarrow \text{SSYT}_{\zeta,\eta}$ induce bijections $\tilde{\mathcal{F}}_{\lambda,\mu} \rightarrow \tilde{\mathcal{D}}_{\lambda,\mu}$ and $\tilde{\mathcal{D}}_{\lambda,\mu} \rightarrow \text{SSYT}_{\lambda,\mu}$ respectively. The following diagram, all of whose squares commute, summarizes our constructions:

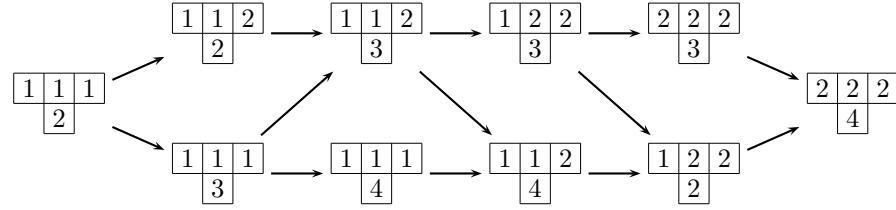
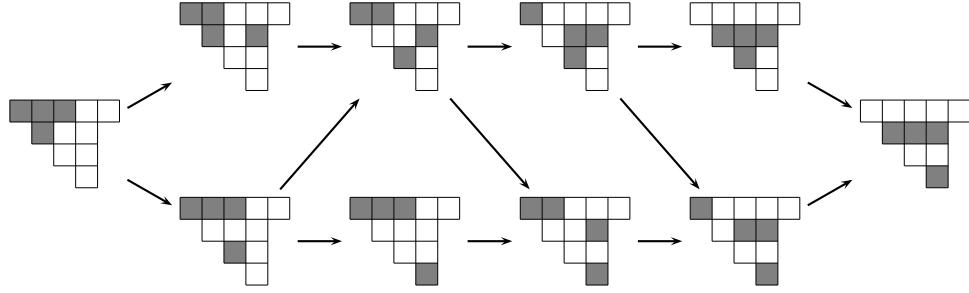
$$\begin{array}{ccccccc}
\mathcal{F}_{\zeta,\eta}'' & \xlongequal{\quad} & \mathcal{F}_{\zeta,\eta} & \longrightarrow & \mathcal{D}_{\zeta,\eta} & \longrightarrow & \text{SSYT}_{\zeta,\eta} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tilde{\mathcal{F}}_{\zeta,\eta}'' & \xlongequal{\quad} & \tilde{\mathcal{F}}_{\zeta,\eta} & \longrightarrow & \overline{\mathcal{D}}_{\zeta,\eta} & \longrightarrow & \text{SSYT}_{\zeta,\eta} \\
\rho \downarrow & & \rho \downarrow & & \downarrow \rho & & \downarrow \rho \\
\tilde{\mathcal{F}}_{\lambda,\mu}'' & \xlongequal{\quad} & \tilde{\mathcal{F}}_{\lambda,\mu} & \longrightarrow & \tilde{\mathcal{D}}_{\lambda,\mu} & \longrightarrow & \text{SSYT}_{\lambda,\mu}
\end{array}$$

All horizontal maps are bijections, as are the four lower vertical maps. Here we are interested in the bottom row, which gives four equivalent combinatorial models.

The families $\mathcal{F}_{\zeta,\eta}''$ appear in [Ko–Ra], [Kra1], [Kra2], [Kre1], and [Kre2]; $\mathcal{F}_{\zeta,\eta}$, $\mathcal{D}_{\zeta,\eta}$, and $\text{SSYT}_{\zeta,\eta}$ appear in [Kre3]; $\tilde{\mathcal{F}}_{\zeta,\eta}''$ and $\tilde{\mathcal{F}}_{\lambda,\mu}''$ were introduced in [Gh–Ra]; $\mathcal{D}_{\zeta,\eta}$, $\overline{\mathcal{D}}_{\zeta,\eta}$, and $\tilde{\mathcal{D}}_{\lambda,\mu}$ were discovered independently by Ikeda–Naruse.

Example 5.1. Let $\lambda = (3, 1)$, $\mu = (5, 3, 2, 1)$. Below we give all elements of $\tilde{\mathcal{F}}_{\lambda,\mu}$, $\tilde{\mathcal{D}}_{\lambda,\mu}$, and $\text{SSYT}_{\lambda,\mu}$.





Proof of Lemma 4.2. Let $\lambda = \sigma(\alpha)$, $\mu = \sigma(\beta)$, $\eta = \pi(\beta)$. Recall that the coordinates $y_{a,b}$ on \mathcal{O}_β are indexed by $\mathfrak{R}_\beta = \{(a,b) \in \beta' \times \beta \mid a \leq \bar{b}\}$. Let $\{v_{a,b} \mid (a,b) \in \mathfrak{R}_\beta\} \subset \mathcal{O}_\beta$ denote the basis dual to the basis of linear forms $\{y_{a,b} \mid (a,b) \in \mathfrak{R}_\beta\} \subset \mathcal{O}_\beta^*$. For $F \in \tilde{\mathcal{F}}''_{\lambda,\mu}$, define

$$W_F = \text{Span}(\{v_{\beta'(x), \beta(n+1-z)} \mid (x,z) \in \text{Supp}(F)\} \dot{\cup} \{v_{a,b} \mid (a,b) \in \mathfrak{R}_\beta, a > b\}).$$

In [Gh–Ra], an explicit equivariant bijection is constructed from

$\mathbb{C} \left[\bigcup_{F \in \tilde{\mathcal{F}}''_{\lambda,\mu}} W_F \right]$ to $\mathbb{C}[Y_{\alpha,\beta}]$. Thus

$$\text{Char}(\mathbb{C}[Y_{\alpha,\beta}]) = \text{Char} \left(\mathbb{C} \left[\bigcup_{F \in \tilde{\mathcal{F}}''_{\lambda,\mu}} W_F \right] \right) = \text{Char} \left(\mathbb{C} \left[\bigcup_{F \in \tilde{\mathcal{F}}_{\lambda,\mu}} W_F \right] \right). \quad (3)$$

Let \tilde{D}_μ be the shifted diagram associated to μ . For $x, z \in \{1, \dots, n\}$, $x \leq z$, we have that $(x, z) \in \tilde{D}_\mu \iff z \leq \eta_x \iff \beta'(z) < \overline{\beta'(x)} \iff \beta'(x) < \overline{\beta'(z)} = \beta(n+1-z)$ (see proof of Remark 3.3). Thus $\{(a,b) \in \mathfrak{R}_\beta \mid a < b\}$ can be expressed as $\{(\beta'(x), \beta(n+1-z)) \mid (x,z) \in \tilde{D}_\mu\}$. Let $F \in \tilde{\mathcal{F}}_{\lambda,\mu}$, and let $D \in \tilde{\mathcal{D}}_{\lambda,\mu}$ and $P \in \text{SSYT}_{\lambda,\mu}$ correspond to F under the bijections above. Since $\text{Supp}(F)$ and D are complements in \tilde{D}_μ ,

$$\begin{aligned} \mathfrak{R}_\beta = & \{(\beta'(x), \beta(n+1-z)) \mid (x,z) \in \text{Supp}(F)\} \dot{\cup} \{(a,b) \in \mathfrak{R}_\beta \mid a > b\} \\ & \dot{\cup} \{(\beta'(x), \beta(n+1-z)) \mid (x,z) \in D\}. \end{aligned}$$

Therefore

$$\begin{aligned}
W_F &= V(\{y_{\beta'(x), \beta(n+1-z)} \mid (x, z) \in D\}) \\
&= V(\{y_{\beta'(x), \beta(n+1-x+r(x)-c(x))} \mid x \in P\}) \\
&= V(\{y_{\beta'(x), \beta(n+1-z(x))} \mid x \in P\}) \\
&= V(\{y_{\beta'(x), \overline{\beta'(z(x))}} \mid x \in P\}) \\
&= W_P.
\end{aligned}$$

Consequently, $\bigcup_{F \in \tilde{\mathcal{F}}_{\lambda, \mu}} W_F = \bigcup_{P \in \text{SSYT}_{\lambda, \mu}} W_P$, and thus

$$\text{Char} \left(\mathbb{C} \left[\bigcup_{F \in \tilde{\mathcal{F}}_{\lambda, \mu}} W_F \right] \right) = \text{Char} \left(\mathbb{C} \left[\bigcup_{P \in \text{SSYT}_{\lambda, \mu}} W_P \right] \right) = \text{Char}(\mathbb{C}[W_{\alpha, \beta}]). \quad (4)$$

Combining (3) and (4), we obtain $\text{Char}(\mathbb{C}[Y_{\alpha, \beta}]) = \text{Char}(\mathbb{C}[W_{\alpha, \beta}])$. By (4) of [Kre3], $[Y_{\alpha, \beta}]_{\mathbb{K}} = [W_{\alpha, \beta}]_{\mathbb{K}}$. \square

References

- [Bil] S. Billey, *Kostant polynomials and the cohomology ring for G/B* , Proc. Nat. Acad. Sci. U.S.A. **94** (1997), no. 1, 29–32.
- [Buc] A. Buch, *A Littlewood-Richardson rule for the K -theory of Grassmannians*, Acta Math. **189** (2002), no. 1, 37–78.
- [Gh–Ra] S. Ghorpade and K. N. Raghavan, *Hilbert functions of points on Schubert varieties in the symplectic Grassmannian*, preprint, arXiv:math.RT/0409338.
- [Gr–Ra] S. Griffeth and A. Ram, *Affine Hecke algebras and the Schubert calculus*, European J. Combin. **25** (2004), no. 8, 1263–1283.
- [Gra] W. Graham, *Positivity in equivariant Schubert calculus*, Duke Math. J. **109** (2001), no. 3, 599–614.
- [Ike] T. Ikeda, *Schubert classes in the equivariant cohomology of the Lagrangian Grassmannian*, preprint, arXiv:math.AG/0508110.
- [KMY1] A. Knutson, E. Miller, and A. Yong, *Gröbner geometry of vertex decompositions and of flagged tableaux*, preprint, arXiv:math.AG/0502144.
- [KMY2] ———, *Tableau complexes*, preprint, arXiv:math.CO/0510487.
- [Ko–Ra] V. Kodiyalam and K. N. Raghavan, *Hilbert functions of points on Schubert varieties in Grassmannians*, J. Algebra **270** (2003), no. 1, 28–54.

- [Kra1] C. Krattenthaler, *On multiplicities of points on Schubert varieties in Grassmannians*, Sémin. Lothar. Combin. **45** (2000/01), Art. B45c, 11 pp. (electronic).
- [Kra2] ———, *On multiplicities of points on Schubert varieties in Grassmannians II*, J. Algebraic Combin. **22** (2005), 273–288.
- [Kre1] V. Kreiman, *Monomial bases and applications for Richardson and Schubert varieties in ordinary and affine Grassmannians*, Ph.D. thesis, Northeastern University, 2003.
- [Kre2] ———, *Local properties of Richardson varieties in the Grassmannian via a bounded Robinson-Schensted-Knuth correspondence*, preprint, arXiv:math.AG/0511695.
- [Kre3] ———, *Schubert classes in the equivariant K-theory and equivariant cohomology of the Grassmannian*, preprint, arXiv:math.AG/0512204.
- [Wil] M. Willems, *K-théorie équivariante des variétés de drapeaux et des variétés de Bott-Samelson*, preprint, arXiv:math.AG/0204265.

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July 9, 2006